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Transverse instabilities for multiple nonrigid bunches in a storage ring

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We describe a method for determining the stability of a system consisting of several highly relativistic bunches of charged particles circulating in a storage ring. The particles interact with magnets designed to guide the beam as well as with electromagnetic fields induced by the particles themselves. Previous calculations considered multibunch modes with one type of internal motion; our method includes coupling between these modes. We also include effects of feedback systems designed to correct these dipole motions. We include an example from a real storage ring design.

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We consider a storage ring in which there are several bunches of charged particles circulating at a frequency ω_0. The charged particles are confined so as to keep them bunched and moving near an ideal orbit. The charges also induce electromagnetic fields (wakefields) which can then act back on the charges, potentially causing the beam to be unstable. These instabilities can often be corrected by making a feedback system which senses the dipole moment of the beam at a given position and acts back on the beam at a different position.

Previous work has considered several aspects of this effect. We will confine our description to work considering the force due to transverse wakefields. Authors have considered how the various eigenmodes involving transverse and longitudinal motion in a single bunch can couple together and create instability (transverse mode coupling) [1–6]. Authors have also shown how to calculate multibunch eigenfrequencies where the bunches are considered to be point particles [7] or where internal degrees of freedom are included but are not considered to be coupled [8,9].

Here, we show how to compute eigenfrequencies, and thus determine the stability, for modes of multiple bunches where the internal degrees of freedom are considered and allowed to couple. The technique we use here is a straightforward extension of the formalism of Wang [8]. The single-bunch case for this has been worked out by Ruth [3,4].

We will describe the force that a single particle feels by a Hamiltonian [10]. The Hamiltonian is written with s, the position along an ideal orbit through the ring, as the independent variable. Because of the bunched nature of the beam, the Hamiltonian may be written as a sum of the Hamiltonians H_r that describe each bunch. Each bunch Hamiltonian will be the sum of two terms.

The first term describes the forces due to magnets, as well as any "potential well distortion" due to longitudinal wakefield forces [11,12], which may be different for different bunches. We assume that the motion due to these forces is at least approximately integrable, so we can write this term as a function of the action variables only [10]. We call this term $H_{r0}(\boldsymbol{J})$, where r is the bunch index.

The second term in the bunch Hamiltonian describes the forces due to wakefields generated by the bunches. The fields induced at a position s in the ring are proportional to the transverse displacement and charge of a particle passing that position. The transverse force a time τ later at the same position in the ring will then be proportional to this dipole moment of the source charge times a transverse wakefield $W_\perp(\tau)$. Taking into account the fact that a storage ring is periodic with length L, the second term in the Hamiltonian for the rth bunch is

$$-\frac{r_0}{\beta_0^2\gamma_0 L}\, y_r(\boldsymbol{\theta},\boldsymbol{J})\sum_{kn} N_n \int y_n(\boldsymbol{\theta}',\boldsymbol{J}')\Psi_n(\boldsymbol{\theta}',\boldsymbol{J}',s-kL) \times W_\perp[\tau_r(\boldsymbol{\theta},\boldsymbol{J})-\tau_n(\boldsymbol{\theta}',\boldsymbol{J}')+kT_0]\,d^2\boldsymbol{\theta}'\,d^2\boldsymbol{J}'. \tag{1}$$

Here r_0 is the classical radius of the electron; β_0 is the average velocity of a reference particle divided by the speed of light; $\gamma_0=(1-\beta_0^2)^{-1/2}$; $\beta_0 c T_0=L$ where c is the speed of light; N_n is the number of particles in bunch n; $y_r(\boldsymbol{\theta},\boldsymbol{J})$ is the transverse displacement for particles in bunch r; $\tau_r(\boldsymbol{\theta},\boldsymbol{J})$ is the difference between the arrival time of a particle in bunch r and the arrival time of a synchronous particle; and $\Psi_n(\boldsymbol{\theta},\boldsymbol{J},s)$ is the phase space distribution function for bunch n at a position s. Note that y_r and τ_r represent two of the four phase space coordinates, and these functions describe part of a canonical transformation to the coordinates y and τ from the action-angle coordinates $\boldsymbol{\theta}$ and $\boldsymbol{J}$.

We have assumed here that the ring is uniform, in the following senses: First of all, the transverse displacement y_r and time-of-flight displacement τ_r, written in terms of action-angle variables, are independent of s. Secondly, $W_\perp$ is also independent of s. By doing this, we assume that coherent synchrobetatron resonances are avoided [5,11,13,14], which turns out to be a good approximation [14]. We correct for nonconstant lattice functions by weighting each piece of the wakefield by the average of the β function over the source of the wakefield [2,13–15].

We can also consider the effect of adding a feedback system through an extra term in the Hamiltonian similar to Eq. (1) above. Simply replace kL in that equation with $kL+\Delta s$ (and thus kT_0 by $kT_0+\Delta s/\beta_0 c$), and rename W to $W_{\rm FB}$. We have once again assumed uniformity around the ring, and are thus ignoring coherent synchrobetatron resonance effects [5,11,13,14], which should be straightforward to avoid.

We now consider a distribution $\Psi_n'(\boldsymbol{\theta},\boldsymbol{J},s)=\Psi_{n0}(\boldsymbol{J})+\Psi_n(\boldsymbol{\theta},\boldsymbol{J},s)$, where Ψ_n is a small perturbation to Ψ_{n0}. We further assume that

$$\int y\Psi_{n0}[J(y,p_y,\tau,p_\tau)]dy\ dp_y\ dp_z=0, \qquad (2)$$

where y and τ are the canonical phase space coordinates corresponding to the transformation described above, and p_y and p_τ are the corresponding canonical momenta. If Eq. (2) were not true (it generally is true in the vertical direction), we have a transverse potential-well distortion problem similar to the longitudinal case [11,12].

The time evolution of the distribution Ψ of particles where the motion of any individual particle is described by a Hamiltonian H is given by the Vlasov equation [11]

$$\frac{\partial\Psi}{\partial s}+[\Psi,H]=0, \qquad (3)$$

where [ ] are Poisson brackets [10].

Because of the bunched nature of the beam, we may write an individual Vlasov equation for each bunch. The Vlasov equations will end up being coupled through the wakefield term in each Hamiltonian. The Vlasov equation for bunch r is

$$\frac{\partial\Psi_r'}{\partial s}+[\Psi_r',H_r]=0. \qquad (4)$$

Note that Ψ_{r0} is independent of s. Second, note that from Eq. (2), H_r does not contain Ψ_{n0}. We will also ignore terms that are second order in Ψ_r.

It is useful to define the impedance, which is the Fourier transform of the wakefield:

$$Z_\perp(\omega)=i\int_{-\infty}^{\infty}W_\perp(\tau)e^{i\omega\tau}d\tau. \qquad (5)$$

Next we define the Fourier transform of the perturbation to the bunch distribution:

$$\Psi_r(\boldsymbol{\theta},\boldsymbol{J},\Omega)=\frac{1}{\beta_0 c}\int \Psi_r(\boldsymbol{\theta},\boldsymbol{J},s)e^{i\Omega s/\beta_0 c}ds. \qquad (6)$$

Finally, it is useful to define the dipole moment due to the entire train of bunches seen in the laboratory frame at frequency $p\omega_0+\Omega$ as

$$D_p(\Omega)=\sum_r N_r\int e^{i(p\omega_0+\Omega)\tau_r(\boldsymbol{\theta},\boldsymbol{J})}y_r(\boldsymbol{\theta},\boldsymbol{J})\Psi_r(\boldsymbol{\theta},\boldsymbol{J},\Omega)d^2\boldsymbol{\theta}d^2\boldsymbol{J}. \qquad (7)$$

The Vlasov equation then becomes

$$\left[\Omega+i\boldsymbol{\omega}_r(\boldsymbol{J})\cdot\frac{\partial}{\partial\boldsymbol{\theta}}\right]\Psi_r(\boldsymbol{\theta},\boldsymbol{J},\Omega) =-\frac{r_0c^2}{\gamma_0L^2}\sum_p D_p(\Omega)Z_\perp(\Omega+p\omega_0) \times\frac{\partial\Psi_{r0}}{\partial\boldsymbol{J}}\cdot\frac{\partial}{\partial\boldsymbol{\theta}}[y_r(\boldsymbol{\theta},\boldsymbol{J})e^{-i(p\omega_0+\Omega)\tau_r(\boldsymbol{\theta},\boldsymbol{J})}], \qquad (8)$$

where $\boldsymbol{\omega}_r=\beta_0c(\partial H_{r0}/\partial\boldsymbol{J})$. A solution of this equation with $\text{Im}\{\Omega\}>0$ corresponds to the existence of an unstable mode.

Adding a feedback system simply adds a term to the right hand side of Eq. (8), with $Z_\perp(\Omega+p\omega_0)$ replaced by $Z_{\text{FB}}(\Omega+p\omega_0)e^{-ip\Delta s/R}$. Z_{FB} is the Fourier transform of W_{FB} as in Eq. (5).

To find the eigenfrequencies Ω in Eq. (8), the equation must be transformed to eliminate Ψ_r and replace it with D_q. To do this, we need to invert the operator

$$\Omega+i\boldsymbol{\omega}_r(\boldsymbol{J})\cdot\frac{\partial}{\partial\boldsymbol{\theta}}. \qquad (9)$$

After doing so, and applying Eq. (7), we get an eigenvalue equation $D_q(\Omega)=\Sigma_pA_{qp}(\Omega)D_p(\Omega)$.

We now assume that we have a lattice consisting only of linear focusing and defocusing magnets. This ignores potential damping effects due to chromaticity [1–4,7–9,11] and tune shift with amplitude [7,16,17].

We also assume that all bunches are identical and equally spaced. If the bunches are all equally spaced but have different numbers of particles (including zero), we can get worst-case values for the growth rates for the symmetric system by setting the single bunch current equal to the current in the highest-current bunch in the real system [18]. Nonequally spaced bunches will only introduce small corrections to the results, since the deviation from equal spacing is typically kept under a few percent [19]. Nonidentical bunches will also cause the synchrotron frequencies of different bunches to be different due to potential well distortion [11,12,20]; this can give an additional damping effect similar to Landau damping [20].

We next assume that the distribution $\Psi_{r0}(\boldsymbol{J})$ depends on $\boldsymbol{J}$ only in the Hamiltonian H_{r0}, and is Gaussian in τ and y. The Gaussian distribution in τ is expected in electron storage rings [21]. Since the longitudinal force due to the transverse wakes is typically small enough to be ignored [3,8,11,13,14,17], we need not be concerned with the precise form of the transverse dependence in Ψ_{r0}.

Since all the bunches are identical, all the r indices in the above equations disappear, except in τ_r. The linear lattice leads to $\boldsymbol{\omega}$ being independent of $\boldsymbol{J}$. Thus, we get

$$\Psi_{r0}(\boldsymbol{J})=\left|\frac{\omega_y\alpha_c^2\beta_0^2c^2}{4\pi^2\sigma_l^4\omega_z^3}\right|e^{-(\omega_yJ_y+\omega_zJ_z)\alpha_c\beta_0c/\omega_z^2\sigma_l^2}, \qquad (10)$$

$$y_r(\boldsymbol{\theta},\boldsymbol{J})=\sqrt{2J_y\beta_y}\cos\theta_y, \qquad (11)$$

$$\tau_r(\boldsymbol{\theta},\boldsymbol{J})=\frac{2\pi r}{M\omega_0}+\left(\frac{2J_z\alpha_c}{\beta_0c\omega_z}\right)^{1/2}\cos\theta_z, \qquad (12)$$

where α_c is the momentum compaction factor, β_y is the average betatron function [22], σ_l is the rms bunch length, and M is the number of bunches. Also, ω_{ry} and ω_{rz} are the y and τ components of $\boldsymbol{\omega}_r$, respectively.

There is a summation over bunch number in the matrix elements A_{qp}. Because the bunches are symmetric, this summation can be performed; it gives zero if $p-q$ is not an integer multiple of M, and M if it is. Thus, we get M separate eigenvalue systems, indexed by p_0, with $p=p_0+\alpha M$ and $q=p_0+\beta M$. Thus, α and β are now the indices on A. The resulting A is

$$A_{\beta\alpha}^{p_0}(\Omega)=i\,\frac{r_0c^2\beta_yNM}{4\gamma_0L^2\omega_y}\,Z_\perp(p\omega_0+\Omega)$$
$$\times e^{-\sigma_l^2[(q\omega_0+\Omega)^2+(p\omega_0+\Omega)^2]/2\beta_0^2c^2}$$
$$\times\sum_m(\delta_{m1}+\delta_{m,-1})\Bigg[2e^{\sigma_l^2(q\omega_0+\Omega)(p\omega_0+\Omega)/\beta_0^2c^2}$$
$$-\frac{\Omega}{\omega_z}\csc\pi\,\frac{\Omega-m\omega_y}{\omega_z}\int_{-\pi}^{\pi}\cos\frac{(\Omega-m\omega_y)\theta'}{\omega_z}$$
$$\times e^{-\sigma_l^2(q\omega_0+\Omega)(p\omega_0+\Omega)\cos\theta'/\beta_0^2c^2}d\theta'\Bigg]. \qquad (13)$$

We take the series for small σ_l of the bracketed term in Eq. (13). Next, we make a change of basis to

$$\phi_n(\Omega)=\left(\frac{\sigma_l}{\beta_0c}\right)^n\sum_\alpha Z_\perp[\Omega+(p_0+M\alpha)\omega_0]$$
$$\times D_{p_0+M\alpha}(\Omega)\;e^{-\sigma_l^2[\Omega+(p_0+M\alpha)\omega_0]^2/2\beta_0^2c^2}$$
$$\times[\Omega+(p_0+M\alpha)\omega_0]^n. \qquad (14)$$

Our system then becomes

$$\phi_m(\Omega)=\sum_{n=0}^{\infty}K_{m+n}(\Omega+p_0\omega_0)F_n(\Omega;\omega_y,\omega_z)\phi_n(\Omega), \qquad (15)$$

$$F_n(\Omega;\omega_y,\omega_z)=\frac{1}{2^nn!}\sum_{k=0}^{n}\binom{n}{k}\frac{[\omega_y+(n-2k)\omega_z]^2}{\Omega^2-[\omega_y+(n-2k)\omega_z]^2}, \qquad (16)$$

$$K_k(\omega)=-i\,\frac{r_0c^2\beta_yNM}{\gamma_0L^2\omega_y}\sum_\alpha Z_\perp(\omega+M\alpha\omega_0)$$
$$\times e^{-\sigma_l^2(\omega+M\alpha\omega_0)^2/\beta_0^2c^2}\left(\frac{\sigma_l}{\beta_0c}\right)^k(\omega+M\alpha\omega_0)^k. \qquad (17)$$

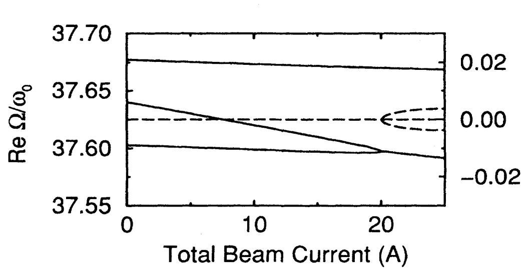

FIG. 1. Single-bunch mode coupling. Solid lines are the real part of the mode frequencies, dashed lines are the imaginary parts (growth rates). Of the real parts, the center line is the $m=0$ mode, while the top and bottom lines are the $m=1$ modes.

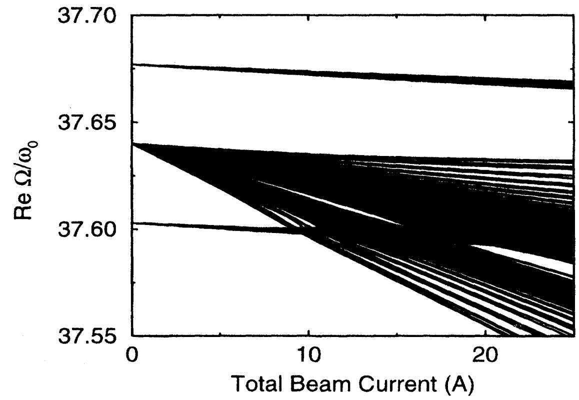

FIG. 2. Multibunch modes with mode coupling, real part of frequencies. Shown are three groups of 1746 lines, each line corresponding to a multibunch mode frequency. The center group are the $m=0$ modes, the top and bottom groups are the $m=1$ modes. Notice that for some of the multibunch modes, the $m=0$ and $m=1$ modes have the same frequency at 10 A.

As an example, we consider the PEP-II B Factory [23] low-energy ring. We will only calculate the effects involving the $m=0$ and $m=1$ terms in Eq. (15). If we consider only a single bunch, we find that the $m=0$ and $m=1$ modes couple, as expected (Fig. 1). If we consider multiple bunches, but do not consider coupling between the modes, we would see growth rates and frequency shifts approximately proportional to current. If we now allow the multibunch modes to couple, we see a strong mode coupling effect, at currents much smaller than the mode coupling threshold for a single bunch (Fig. 2; compare Fig. 1). The growth rates of the $m=0$ modes which have large growth rates when mode coupling is ignored are only slightly affected by mode coupling. Those which had small growth rates essentially mirror the single bunch behavior of Fig. 1.

The major effect is on the $m=1$ modes. If mode coupling is ignored, these typically have small growth rates which increase linearly with current. Mode coupling affects these modes in two ways. First, the $m=1$ growth rates no longer increase linearly with current, and so for even small currents,

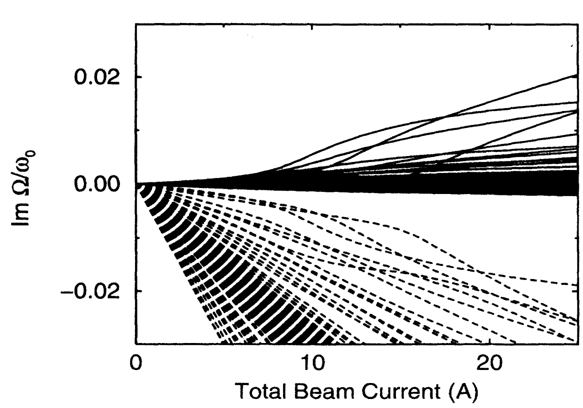

FIG. 3. Multibunch growth rates with feedback. Dashed lines are the $m=0$ modes, other lines arem$=1$.

they can become very large. Second, the sudden increase in growth rate as a function of current that occurs in the single bunch case (see Fig. 1) now occurs at a much lower current. The approximate location of this increase is the point where the real frequencies coincide (see Fig. 2), similar to the single bunch case. In the case of the PEP-II low-energy ring, the operating current will be 3 A at worst, well below the current of 10 A where the multibunch mode frequencies coincide in Fig. 2. Thus, we expect minimal effects from multibunch mode coupling (although the nonlinear increase with current discussed above does have a slight detrimental effect).

Typical feedback systems damp only the $m=0$ modes. Damping the $m=1$ modes requires pickups and kickers that operate at very high frequencies. Thus, the strong effect of mode coupling on the $m=1$ modes can produce instabilities that are difficult to correct, and are at much lower currents than the single-bunch mode coupling threshold (see Fig. 3).

Radiation damping, as well as other damping effects which we have ignored in this calculation, may serve to reduce the growth rates that have been computed by this method. If the growth rates computed here are smaller than the the damping rates from these effects, then we may assume that the beam is stable.

To conclude, we have demonstrated a method for calculating the eigenfrequencies for a storage ring containing multiple bunches, which allows one to include coupling between the internal degrees of freedom in the bunches. We show that this can have an effect similar to single-bunch mode coupling, but at significantly lower currents. This is an important effect to consider in the design of the next generation electron-positron storage rings such as the PEP-II B Factory.

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